# RESEARCH STATEMENT

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My main research interests are: analytic number theory, discrete harmonic analysis, and moments of arithmetic functions. I have completed research projects in ergodic theorem, primes distribution, theory of partitions, and additive number theory. My notable results during my Ph.D. are

- Sharp $\ell^p-$improving inequalities of several averages over prime numbers
- Reporting a cancellation phenomenon as well as its application in partition theory, distribution of prime numbers, and Waring's problem.
- Solving a statistical version of the 105 problem raised by Erdős and Graham.

I propose a few ideas as possible future directions.

Contents



## 1. Future Directions

1.1. **Current Projects.** I am working on two problems in ergodic theory. The first one is to study the averages of the form

$$A_{N,y,b}(x) := \frac{\phi(y)}{N} \sum_{\substack{n \equiv b \pmod y \\ n<N}} \Lambda(n) f(x-n).$$

We are studying the $\ell^p$-improving inequality for $A_{N,y,b}$. The natural bound is

$$\| A_{N,y,b} \|_{\ell^{p'}} \ll \left(\frac{y}{N}\right)^{\frac{1}{p}-\frac{1}{p'}} \| f \|_{\ell^p} \tag{1.1}$$

However, we need to have a very mild assumption on the progression distribution of $f$ modulo $y$.

The second project tat I am working on at the moment is to prove the Carleson averages along the prime numbers. In particular, we are interested in proving that the following operator is $\ell^p$ bounded :

$$\mathcal{C}f(x) := \sup_{\lambda} \sum_{n} \Lambda(n) e\left(P_\lambda(n)\right) \frac{f(x-n)}{n},$$

where $P_\lambda(n) = \lambda_d n^d + \cdots + \lambda_1 n$. Note that to prove this result, we need to use a sharp prime gap result and variants of the exponential bounds.

1.2. **Related to the arithmetic functions in progression.** I am curios to study the sum

$$\frac{1}{H}\sum_{\substack{n\equiv b \pmod y\\ x-H<n<x+H}}\Lambda(n)f(x-n)$$

Note that it is not covered by Krause,Mirek,Tao's bilinear result in [7], because the average is over the primes. As a more advanced question, one can also check to see if the Carleson averages like

$$\frac{1}{H}\sum_{\substack{n\equiv b \pmod y\\ x-H<n<x+H}}\Lambda(n)f(x-n)\frac{e(P(n))}{n}$$

are $\ell^p$ bounded. To do that, a $TT^*$ argument would be helpful, in which we need the variance of the number of squarefree integers in short interval with arithmetic progression.

Also, in [16], there is a similar result for the expectation of a function in arithmetic progression.

$$\left|\mathbf{E}_{n<X,n\equiv a \pmod q}f(n)-\mathbf{E}_{n<X}f(n)\right|<\epsilon \qquad\qquad \text{for multiplicative function } f$$

with the exception of at most $X^{-c\sigma\epsilon}Q\epsilon^{-1}$ for primes $q>X^{1/2}$. I like to check and see if we can prove a result for $f(n)=\Lambda(n)\Lambda(x-n)$ in any possible range of $q$. Moreover, I like to see if we can use this method to prove (1.1) for larger $y$. Due to large height of the multiplier, our proof was very technical even in the case $y<e^{c\sqrt{\log(N)}}$. It would be interesting to prove (1.1) without any assumption over the arithmetic structure of $f$.

1.3. **Related to average inequalities.** There are several directions related to this project, and I mention a few of them. One way is to check the average inequalities in number fields or function fields terminology. I am interested in proving a quantitative inequality for

$$A_Nf(x)=\frac{1}{N^2}\sum_{N(n_1),N(n_2)<N}\Lambda_{\mathbb{F}}(n_1)\Lambda_{\mathbb{F}}(n_2)f(x-n_1-n_2)$$

where $\mathbb{F}$ is a quadratic number field with class number 1. We have a good chance of proving at least an average version of a result for Gaussian integers (with more logarithm weight functions) because it is known that Gaussian primes have uniform distribution both in magnitudes and in-phase. As for the function fields, papers like [17, 18] studied the the averages over irreducible polynomials.

$$E(N,K,q):=\frac{1}{q^N}\sum_{\substack{f\in\mathbb{F}_q[T]\\ \deg(f)=N}}\Lambda(f)\Lambda(f+K).$$

I would like to study norm covergence of such averages. For example if

$$A(N,K,q):=\frac{1}{q^N}\sum_{\substack{f\in\mathbb{F}_q[T]\\ \deg(f)=N}}\Lambda(f)T(f+K)$$

Then can we prove inequalities like

$$\sum_{\substack{K\in\mathbb{F}_q[T]\\ \deg K=N}}A(N,K,q)^2\ll C(N,q)\sum_{\substack{f\in\mathbb{F}_q[T]\\ \deg(f)=N}}T^2(f) \text{ for some } C(N,q)>0?$$

I am also interested in studying the bilinear averages along the primes as follows:

$$A_Nfg(x):=\frac{1}{N}\sum_{n<N}\Lambda(n)f(x-n)g(x-n^2).$$

Proving any pointwise convergence or norm convergence can be interesting for the people in the subject. Aside from computing the uniform norm of von Mangoldt function and using the average estimates on prime gaps, we need to control *bilinear Kloosterman's sum.* As a toy example, We have to deal with the one-dimensional case for Ramanujan's sum:

$$\sum_{t\in\mathbb{Z}/\mathbb{Z}_q}\mathbf{1}_{t\in S}c_q(x+t)$$

where $S$ can be sets related to the nature of the bilinear average. The bilinear arguments will end up in studying

$$\sum_t \mathbf{1}_{t_1\in S_1}\mathbf{1}_{t_2\in S_2}K_q(f(t_1,t_2);q) \text{ for some function } f.$$

I also like the results involving the Vinogradov theorem with almost equal summands. I believe that it can be generalized in the context of ergodic theory, and I like to work on that. For example, one can study the convergence of the averages like

$$A(x) := \sum_{u,t<N^\theta} \Lambda(n)\Lambda(n+t)\Lambda(n+t+u)1_{x=3n+u+t}.$$

I like to study different kinds of convergence on $A(x)$. We have to construct multifrequency machinery to control the high part for this average.

In the paper [11], a very sharp bound is computed for

$$S(x) := \sum_{n\leqslant x} f(n) \text{ for some multiplicative function } f.$$

There is vibrant literature involving generalizations of this result for different scenarios (see for example [12]). I believe that these results can be studied from the discrete ergodic theorem point of view. Being multiplicative is extra information about $f$, which may help us control the low part better. One the examples of such averages would be what is the $\ell^p$ norm upper bound for the Carleson's operator

$$Cf(x) = \sup_{\lambda>0}\sum_{n<N} f(n)\frac{e((x-n)\lambda^2)}{n}$$

when $f$ is a multiplicative function?

1.4. **Related to Vinogradov mean value theorem and Waring's problem.** We answered an approximated problem as follows.

**Question 1.2.** *Let $0<c=c(N,n,k)<1$ be the smallest constant such that there exists increasing sequences of positive integers $\{a_i\}$ and $\{b_i\}$ and $a_i,b_i<N$ for $1\leqslant i\leqslant n$ that do not overlap, i.e. $a_i\neq b_j$, such that for all $1\leqslant r\leqslant k$,*

$$\sum_{i=1}^n a_i^r - b_i^r \ll N^{cr}$$

*How small can we take $c$ to be for various ranges of $k$ and $n$?*

We proved that one can choose $k$ to be very large, for example, $n^{1-\epsilon}$ and find a solution for the approximated version of Prouhet-Tarry-Escott. We tried to iterate the same algorithm to shrink the error term all the way to zero. However, the main issue was the fact that our function would not remain analytic. As an easy example, take, for example

$$f(z_1,z_2) := \frac{\left(4x^2 - z_1^2 - \left(x-z_1^2\right)^{2r}\right)^{2r}}{\sin(\pi z_1)\sin(\pi z_2)}$$

gives a summation of powers of $r$, but the problem is that we cannot use the residue theorem after bounding the first integral. I like to check if I can make the error term small enough by making the iteration possible. Then we can apply decoupling or efficient congruency to control the error term and find an actual solution in an out-of-reach range of $k,n$.

1.5. **Moments of Riemann zeta function.** I like working on the moments of arithmetic functions and approximatation formulas for the moments of $L-$functions and Dirichlets series. I believe that our cancellation formula can give a new approximation formula. We can study the following integral:

$$\int_1^T \left|\sum_{n^2<NT} \exp\left(\sigma + 2\pi i t\sqrt{N-n^2/T}\right)\right|^{2k} dt,$$

where $2k\in\mathbb{Q}$ and $\sigma$ should be chosen carefully. Our method is helpful because we can get nontrivial bounds for the low-height zeros.

There are many results about the connection between zeros of different categories of $L-$functions and eigenvalues of random matrices (see for example [14, 15]). The equation (**??**) gives us a very sharp (even sharper than RH) estimate on the number of primes in a set of intervals. Admittedly, the main reason for the cancellation is not rooted from the prime numbers but comes from the structure of exponential functions and polynomials. In other words, I am not sure if we find a piece of new knowledge about prime numbers, but I am curious to find the correspondence phenomenon on the random matrix theory. The main reason is that I believe there exists even more cancellation, and we simply could not detect it. So the direction of my proposed project is like what follows.

Phenomenon $A$ : Cancellation phenomenon on zeros of zeta function

⇒Phenomenon $A'$ : Checking the same kind of cancellation in random matrices

⇒Phenomenon $B'$ : Exploring the phenomenon on random matrices to see a more general or stronger pattern

⇒Phenomenon $B$ : Finding the correspondence of the more general pattern zeta functions

⇒ Attempting to prove it or making it equivalent with known famous conjectures

1.6. **Directions related to the 105 problems.** In [20], the following result has been studied:

$$|\{p < x \text{ such that } s_b(p) \equiv a \pmod{m}\}| = \frac{x}{m\log(x)} + \text{ error terms}$$

where $s_b(p)$ is the sum of digits of $p$ in base $b$. This problem is similar to what we did in 105 problem, and I am interested to study the same question in for example two bases simultanously. For example can we say

$$|\{p < x \text{ such that } s_b(p) \in \mathcal{S}_1 \text{ and } s_c(p) \in \mathcal{S}_2\}| = \frac{x}{ym\log(x)} + \text{ error terms}$$

where $\mathcal{S}_1, \mathcal{S}_2$ are two sets with progression structure module $m$ and $y$? A similar problem can be defined inspiring from [21].

We could not completely prove the 105 problem, but maybe one way to study it would be to change the weight of the digits. In other words, assume that we put weight $w_{i,j}$ to the $i$th block in $j$th prime. Then a good question would be keeping the notation in section 1, for which set of weights we get

$$\sum_i w_{i,j} \times \text{ number of digits greater than } p_j/2 \text{ in block } i \text{ of base } p_j \text{ of } n = O(1). \text{ for every } 1 \leqslant j \leqslant r \tag{1.3}$$

In [19], if for a family of functions we have $v(f) < M$, then for which categories we get $v(f) \simeq M$. What we ar interested is to find out that for which set of weights $\{w_{i,j}\}$ we get (1.3).

The main result in the paper [10] is to introduce more than 50 new irreducible forms like

$$\frac{(a_1n)!(a_2n)!\cdots(a_kn)!}{(b_1n)!(b_2n)!\cdots(b_{k+2}n)!} \tag{1.4}$$

that are integers. In our project to study 105 problem, we could formulate the condition $p_i \parallel \frac{(2n)!}{n!n!}$ as an additive problem. Although the goals of the problems are different, I would guess that there might be ways to formulate the condition where all the prime exponents of forms like in (1.4) become positive. Once we get an additive problem, we can discuss it statistically to find possible limitations in identifying these criteria. I believe it is a viable research project.

1.7. **Finding a way to attack our conjecture on cancellation phenomenon.** Numerical results suggest that

$$\sum_{\ell^2<x}(-1)^\ell e^{\sqrt{x-\ell^2}} = e^{o(\sqrt{x})}.$$

I am interested in studying this sum to see if we can get a better result. Our limitation in using the circle method was the fact that the vertical legs of contour in [3] can get a significant contribution, and we do not have the exponential explosion in the denominator anymore. I am thinking of using a sharper function like the gamma function to use the residue theorem.

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